Discovery of Non-Euclidean Geometry

April 24, 2013

1 Hyperbolic geometry

János Bolyai (1802-1860), Carl Friedrich Gauss (1777-1855), and Nikolai Ivanovich Lobachevsky (1792-1856) are three founders of non-Euclidean geometry.

Hyperbolic geometry is, by definition, the geometry that assume all the axioms for neutral geometry and replace Hilbert's parallel postulate by its negation, which is called the **hyperbolic axiom**.

Hyperbolic axiom (Negation of Hilbert axiom). There exists a line l and a point P not on l such that at least two distinct lines parallel to l pass through P.

Theorem 1.1. *In hyperbolic geometry, all triangles have angle sum less than* $180°$*, and all convex quadrilaterals have angle sum less than* $360°$*. In particular, there is no rectangle.*

Proof. Trivial. $\square$

Theorem 1.2 (Universal Hyperbolic Theorem)**.** *In hyperbolic geometry, for every line l and every point P not on l there pass through P at least two distinct lines parallel to l. In fact there are infinitely many lines parallel to l through P.*

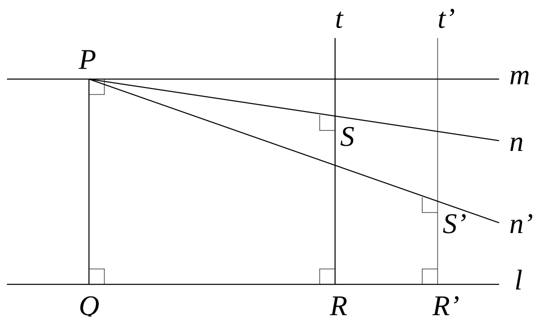

Figure 1: Existence of infinite parallels

Proof. Drop segment PQ perpendicular to l with foot Q on l. Erect line m at P perpendicular to $\overline{PQ}$. Then l, m are parallel. Pick a point R on l other than Q, and erect line t at R perpendicular to l. Drop line n through P perpendicular t, intersecting t at S. If S is on m, then S is the intersection m and t, and subsequently $\square PQRS$ is a rectangle, which is impossible in hyperbolic geometry. So point S is not on m. Hence m, n are distinct lines through P, both are parallel to l. See Figure 1.

Let R' be point on l other than Q, R, and t' be line through R' perpendicular to l. There exists line n' through P perpendicular to t', intersecting t' at S'. If $\overline{PS} = \overline{PS'}$, then $\square RR'S'S$ is a rectangle, which is impossible. So $\overline{PS}, \overline{PS'}$ are distinct lines. Thus for all points R on l other than Q, the lines $\overline{PS}$ through P perpendicular to l are all distinct. $\square$

Definition 1. Two triangles are ΔABC and $\Delta A'B'C'$ said to be **similar** if their vertices can be put in on-to-one correspondence so that corresponding angles are congruent, i.e., $\angle A, \angle B, \angle C$ are congruent to angles $\angle A', \angle B', \angle C'$ respectively.

Theorem 1.3 (AAA criterion for congruence of hyperbolic triangles)**.** *In hyperbolic geometry, if two triangles are similar then they are congruent.*

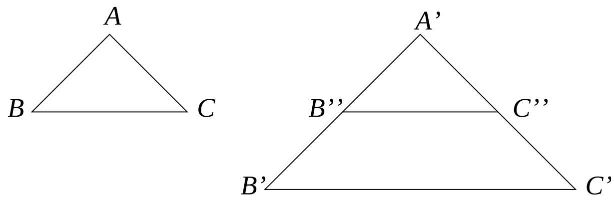

Figure 2: Similar triangles are congruent in hyperbolic geometry

Proof. Given similar triangles ΔABC and $\Delta A'B'C'$. Suppose the statement is not true, i.e., ΔABC is not congruent to $\Delta A'B'C'$. Then $AB \ncong A'B'$, $AC \ncong A'C'$, $BC \ncong B'C'$; otherwise $\Delta ABC \cong \Delta A'B'C'$ by ASA. We may assume $AB < A'B'$ and $AC < A'C'$. Lay off segment AB on ray $r(A', B')$ to have point B'' such that $AB \cong A'B''$ and $A' * B'' * B'$, and lay off segment AC on ray $r(A', C')$ to have point C'' such that $AC \cong A'C''$ and $A' * C'' * C'$. See Figure 2. Then $\Delta ABC \cong \Delta A'B''C''$ by SAS. Hence

$$\angle A'B''C'' \cong \angle B \cong \angle B', \quad \angle A'C''B'' \cong \angle C \cong \angle C'.$$

It follows that lines $\overline{B''C''}, \overline{B'C'}$ are parallel because of Congruent Corresponding Angles. So $\Box B'C'C''B''$ is a convex quadrilateral. Since angles $\angle A'B''C'', \angle B'B''C''$ are supplementary, angles $\angle A'C''B'', \angle C'C''B''$ are supplementary, and $\angle B' \cong \angle A'B''C'', \angle C' \cong \angle A'C''B''$, then the angle sum of $\Box B'C'C''B''$ is $360°$. This is a contradiction. $\Box$

Theorem 1.3 says that in hyperbolic geometry it is impossible to magnify or shrink a triangle without distortion. So in hyperbolic world photography would be inherently surrealistic.

Another consequence of Theorem 1.3 is that the length of a segment may be determined by angles in hyperbolic geometry. For example, an angle of an equilateral triangle determines the length of a side uniquely. This fact is sometimes referred to that hyperbolic geometry has an **absolute unit length**.

2 Parallels that admit a common perpendicular

Given lines l, l' and points $A, B, C, \ldots$ on l. Drop perpendiculars $AA', BB', CC', \ldots$ from $A, B, C, \ldots$ to l' with feet $A', B', C', \ldots$ on l' respectively. We say that $A, B, C, \ldots$ are **equidistant** from l' if all these perpendicular segments are congruent to one another. See Figure 3.

Theorem 2.1 (At most two points equidistant)**.** *Given two distinct parallels l, l' in hyperbolic geometry. Then any set of points on l equidistant from l' contains at most two points.*

Proof. Suppose it is not true, i.e., there is a set of three points A, B, C on l equidistant from l'. Then quadrilaterals $\Box ABB'A', \Box ACC'A', \Box BCC'B'$ are Saccheri quadrilaterals (the base

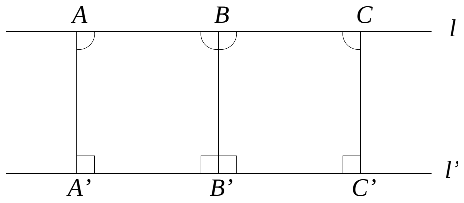

Figure 3: No more than two equidistant points between two parallels

angles are right angles and the sides are congruent). Then the summit angles of the Saccheri quadrilaterals are congruent, i.e.,

$$\angle BAA' \cong \angle ABB', \quad \angle CAA' \cong \angle ACC', \quad \angle CBB' \cong \angle BCC'.$$

Thus $\angle ABB' \cong \angle CBB'$. Since $\angle ABB', \angle CBB'$ are supplementary, they must be right angles. Hence all $\square ABB'A'$, $\square ACC'A'$, $\square BCC'B'$ are rectangles, which is impossible. $\square$

Lemma 2. *Given a Saccheri quadrilateral $\square ABB'A'$ with base right angles $\angle A', \angle B'$ and equal opposite sides AA', BB'. Let M, M' be the middle points of $AB, A'B'$ respectively. Then segment MM' is perpendicular to both lines $\overline{AB}$ and $\overline{A'B'}$.*

Proof. Draw segments $A'M$ and $B'M$. Note that $AA' \cong BB'$, $\angle A \cong \angle B$, and $AM \cong BM$. Then $\Delta A'AM \cong \Delta B'BM$ by SAS. So $A'M \cong B'M$. Hence $\Delta A'MM' \cong \Delta B'MM'$ by SSS. We then have $\angle A'M'M \cong \angle B'M'M$. Subsequently, $\angle A'M'M$ and $\angle B'M'M$ are right angles. So MM' is perpendicular to the base $A'B'$. Note that $\angle A'MM' \cong \angle B'MM'$ and

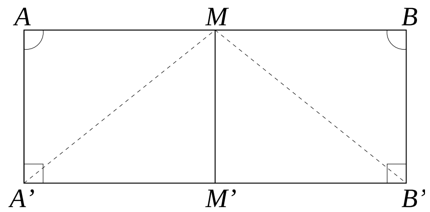

Figure 4: Perpendicular middle point segment

$\angle AMA' \cong \angle BMB'$. Then $\angle AMM' \cong \angle BMM'$ by angle addition. Subsequently, $\angle AMM'$ and $\angle BMM'$ are right angles. So MM' is perpendicular to the summit AB. $\square$

Theorem 2.2 (Divergent and symmetric parallels)**.** *Let l, l' be two lines perpendicular to a segment MM' with $M \in l, M' \in l'$.*

(a) Then $|MM'| < |XY'|$ for all $X \in l, Y' \in l'$ with $XY' \neq MM'$.

(b) If M is the middle point of a segment AB on l, then A, B are equidistant from l'.

(c) If $M * B * D$ on l and BB', DD' are segments perpendicular to l' with feet $B', D' \in l'$, then $BB' < DD'$.

Proof. (a) It is clear that $MM' < MY'$ for all Y' on l' with $Y' \neq M'$. Let X be a point on l with $X \neq M$. Let XX' be segment perpendicular to l' with foot X' on l'. Then $\square MM'XX'$ is a Lambert quadrilateral. Thus

$$MM' < XX'$$

by properties of Lambert quadrilaterals. Since $XX' < XY'$ for Y' on l' with $Y' \neq X'$. We see that $MM' < XY'$.

(b) Let AA', BB' be segments perpendicular to l' with $A', B' \in l'$. Draw segments AM' and BM'. Then $\Delta AMM' \cong \Delta BMM'$ by SAS. So $AM' \cong BM'$ and $\angle AM'M \cong \angle BM'M$.

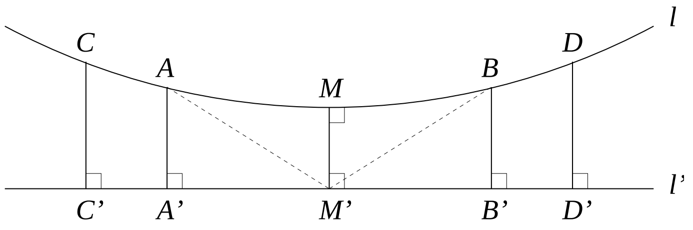

Figure 5: Divergent parallels are symmetric

Subsequently, $\angle AM'A' \cong \angle BM'B'$ by angle subtraction. Thus $\Delta AA'M' \cong \Delta BB'M'$ by SAA. Hence $AA' \cong BB'$ and $A'M' \cong B'M'$.

(c) Note that $\square M'B'BM$ and $\square M'D'DM$ are Lambert quadrilaterals. Then $\angle B'BM$ and $\angle D'DM$ are acute angles. So $\angle B'BD$ is obtuse, for it is supplementary to $\angle B'BM$. Hence $\angle D'DB = \angle D'DM < \angle B'BD$. Therefore $BB' < DD'$ by the property of $\square B'D'DB$. $\square$

Proposition 2.3 (Asymptotic and monotonic parallels)**.** *Given parallels l, l' in hyperbolic geometry, no two points of l are equidistant from l'. Let AA', BB', CC' be perpendicular segments to l' with $A * B * C$ on l and $A', B', C' \in l'$. See Figure 6.*

(a) If $AA' < BB'$, then $BB' < CC'$.

(b) If $BB' < CC'$, then $AA' < BB'$.

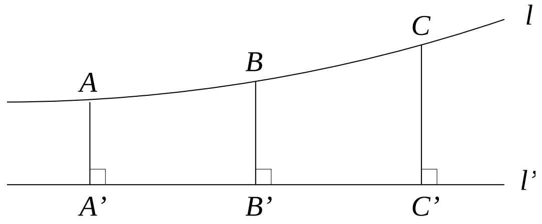

Figure 6: Monotone distance between asymptotic parallels

Proof. Consider quadrilaterals $\square A'B'BA$ and $\square B'C'CB$.

(a) Since $AA' < BB'$, then $\angle ABB' < \angle BAA'$. Since the angle sum of $\square A'B'BA$ is less than $360°$, it follows that $\angle ABB'$ is acute. So $\angle CBB'$ is obtuse. Hence $\angle BCC'$ must be acute, since the angle sum of $\square B'C'CB$ is less than $360°$. Of course $\angle BCC' < \angle CBB'$, subsequently, $BB' < CC'$ by the property of $\square B'C'CB$.

(b) Fix a point D on l with $A * B * C * D$. For each X on the open ray $\mathring{r}(D, A)$ we write $|DX| = x$ and define $f(x) = |XX'|$, where XX' is perpendicular to l' with foot X' on l'. We claim that $f(x)$ is a continuous function for $x > 0$. In fact, fix an x_0 with point X_0 on l such that $|BX_0| = x_0$. Let X_0X_0' be segment perpendicular to l' with $X_0' \in l'$. Note that

$$|XX'| \leq |XX_0'| \leq |X_0X_0'| + |XX_0|, \quad |X_0X_0'| \leq |X_0X'| \leq |XX'| + |XX_0|.$$

Then

$$\begin{aligned} |f(x) - f(x_0)| &= \begin{cases} |XX'| - |X_0X_0'| & \text{if } |XX'| \geq |XX_0'| \\ |X_0X_0'| - |XX'| & \text{if } |XX'| < |XX_0'| \end{cases} \\ &\leq |XX_0| = |x - x_0|. \end{aligned}$$

Clearly, $f(x)$ is continuous at x_0. So $f(x)$ is a continuous function for $x > 0$.

Suppose $AA' > BB'$. Note that $AA' \not\cong CC'$. If $|AA'| < |CC'|$, by intermediate value theorem there exists a Y with $B * Y * C$ such that $|YY'| = |AA'|$. Then A, Y are equidistant from l', which is impossible. If $|AA'| > |CC'|$, by intermediate value theorem there exists a point Z with $A * Z * B$ such that $|ZZ'| = |CC'|$. Then C, Z are equidistant from l', which is impossible.

We then must have $AA' < BB'$. □

3 Limiting parallel rays

Given a line l in hyperbolic geometry and a point P not on l. Let m be a line through P parallel to l with left ray $r(P, R)$. Drop perpendicular segment PQ to l with foot Q on l. We consider rays between $r(P, Q)$ and $r(P, R)$, and want to find the critical ray $r(P, X)$, called the **left limiting parallel ray to l through P**, that does not meet l but any ray between $r(P, X)$ and $r(P, Q)$ meets l. Likewise, there is a **right limiting parallel ray to l through P** on the opposite side of $\overline{PQ}$. See Figure 7.

Theorem 3.1. *Given a line l and a point P not on l in hyperbolic geometry. Let PQ be segment perpendicular to l with foot Q on l. Then there exist two non-opposite rays $r(P, X), r(P, X')$ on opposite sides of line $\overline{PQ}$, satisfying the properties:*

(a) Each of rays $r(P, X), r(P, X')$ does not meet l.

(b) A ray $r(P, Y)$ meets l if and only if it is between $r(P, X)$ and $r(P, X')$.

(c) $\angle QPX \cong \angle QPX'$.

Proof. Let m be the line through P perpendicular to PQ. Pick a point R on the left side of m and a point R' on the right side of m separated by P. Draw segments QR and QR'. Then all rays between $r(P, Q)$ and $r(P, R)$ inclusive are represented by $r(P, Y)$ with $Y \in QR$. See Figure 7.

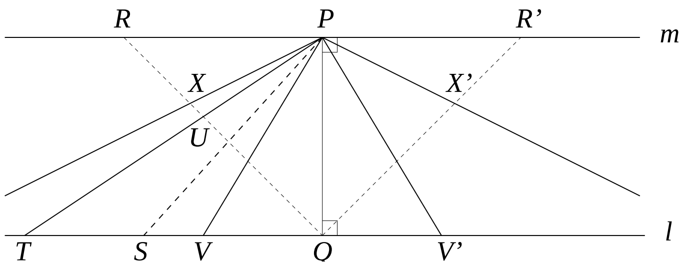

Figure 7: Limiting parallel rays

(a) Let Σ_1 be the set of points $Y \in r(Q, R)$ such that the ray $r(P, Y)$ does not meet l, and Σ_2 the complement of Σ_1 in $\overline{QR}$. It is easy to see that both Σ_1, Σ_2 are convex. So Σ_1, Σ_2 form a Dedekind cut of $\overline{QR}$. Then there exists a unique point $X \in \overline{QR}$ such that Σ_1, Σ_2 are two rays (one of them is an open ray) of $\overline{QR}$ separated by X. We claim that $X \in \Sigma_1$. Suppose $X \in \Sigma_2$, i.e., $r(P, X)$ meets l at S. Pick a point T on l such that $T * S * Q$. Then ray $r(P, T)$ is between $r(P, R)$ and $r(P, X)$. So $r(P, T)$ meets RQ at U and $R * U * X$, i.e., $U \in \Sigma_2$, which is a contradiction. The existence of ray $r(P, X')$ is analogous.

(b) Since $R \in \Sigma_1$ and $Q \in \Sigma_2$, we see that $R * X * Q$. It is obvious that if a ray $r(P, Y)$ is contained in the open half-plane opposite to $\mathring{H}(m, Q)$ then $r(P, Y)$ does not meet l. We then see that a ray $r(P, Y)$ meets l if and only if $r(P, Y)$ is between $r(P, X)$ and $r(P, X')$.

(c) Suppose that $\angle XPQ$ is not congruent to $\angle X'PQ$, say, $\angle XPQ < \angle X'PQ$. Find point V' on l such that $r(P, V')$ is between $r(P, Q)$ and $r(P, X')$, and $\angle QPV' \cong \angle QPX$. Mark a point V on l such that $V * Q * V'$ and $QV \cong QV'$. Then $\Delta PQV \cong \Delta PQV'$ by SAS. So $\angle QPV \cong \angle QPV' \cong \angle QPX$, i.e., $r(P, X)$ meets l at V, which is a contradiction. □

The angle $\angle QPX$ is called the **angle of parallelism** at point P with respect to l, its degree measure is denoted $\Pi(PQ)^\circ$. We have

$$\Pi(PQ)^\circ < 90^\circ.$$

4 Classification of parallels

Theorem 4.1. *Given parallel lines l, l' in hyperbolic geometry.*

(a) If l contains a limiting parallel ray to l', then l, l' are asymptotic parallels.

(b) If l does not contain limiting parallel ray to l', then l, l' are divergent parallels.

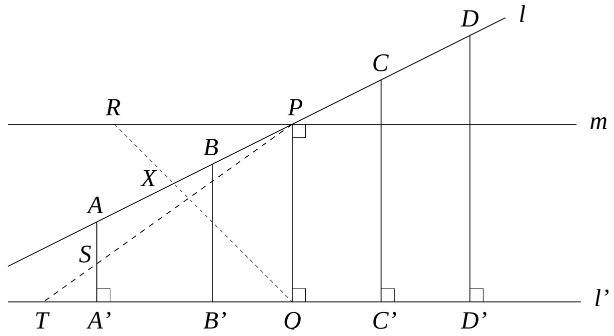

Figure 8: The limiting parallel ray is asymptotic and monotonic

Proof. Fix a point P not on l' and drop a perpendicular PQ to l' with foot $Q \in l'$. Let m be the line through P perpendicular to PQ. Pick a point R on m other than P. Let $r(P, X)$ be a limiting parallel ray to l' with $X \in QR$. See Figure 8.

(a) Let A, B, C, D be points on l with $A * B * P * C * D$ and $A, B \in \mathring{r}(P, X)$. Let AA', BB', CC', DD' be segments perpendicular to l' with feet $A', B', C', D' \in l'$. Note that $\angle XPQ$ is acute, $\angle CPQ$ is obtuse, and the angle sum of $\square PQC'C$ is less than 360°. Then $\angle PCC'$ is acute. Of course $\angle PCC' < \angle CPQ$. So $PQ < CC'$ by property of quadrilaterals with two base right angles.

Analogously, $\angle PDD'$ is acute and $\angle DCC'$ is obtuse. Of course $\angle PDD' < \angle DCC'$. Then $CC' < DD'$ by property of quadrilaterals with two base right angles.

We claim $|AA'| \le |PQ|$ for all A on open ray $\mathring{r}(P, X)$. Suppose $|AA'| > |PQ|$. Let S be a point on AA' such that $|AS| = \frac{1}{2}(|AA'| - |PQ|)$. Clearly, $|SA'| > |PQ|$. Then $\angle A'SP < \angle XPQ$ by property of quadrilateral with two base right angles. Of course $\angle A'SP$ is acute. Since $r(P, S)$ is between $r(P, Q)$ and $r(P, X)$, the ray $r(P, S)$ meets l' at T. Note that $\angle A'ST$ is acute. So $\angle A'SP$ is obtuse, contradicting to that $\angle A'SP$ is acute.

We further claim $AA' < BB'$ for two points A, B on closed ray $r(P, X)$ with $A * B * C$. Suppose $|AA'| \ge |BB'|$. There exists a point E on BP (maybe $B = P$) such that $AA' \cong EE'$ by continuity of distance function. Let M, M' be the middle points of $AE, A'E'$ respectively. Then l, l' are divergent parallels. Let F be on l such that $F * M * C$ and $MF \cong MC$. We have $|FF'| = |CC'| > |PQ|$, which is a contradiction.

(b) Assume that l does not contain any limiting parallel ray. If $l = m$, then l, l' are already divergent parallels. If $l \neq m$, we may assume that a ray $r(P, Y)$ of l is between $r(P, R)$ and $r(P, X)$, where $R * Y * X$. It is easy to see that $PQ < CC' < DD'$ by similar arguments.

Since $\angle XPY$ is acute, by Aristotle's axiom there exists a point A on $r(P, Y)$ such that $AE > PQ$, where AE is perpendicular to $r(P, X)$ with foot $E \in r(P, X)$. Of course $AA' >$

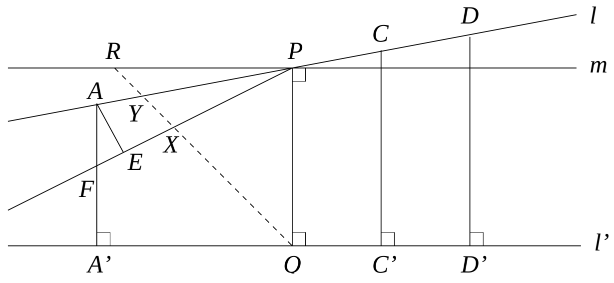

Figure 9: Non-limiting parallel ray is symmetric

$AF > AE$. So $AA' > PQ$. Thus l, l' cannot be asymptotic (monotonic) parallels. So l, l' must be divergent (symmetric) parallels. □

Let A, A' be two distinct points on the same side of a line $\overline{AB}$ such that lines $\overline{AA'}, \overline{BB'}$ are parallel. Then the figure, consisting of the segment AB (called the **base**) and the rays $r(A, A')$ and $r(B, B')$ (called the **sides**), is called a **biangle** with **vertices** A and B, denoted $\angle A'ABB'$. See Figure 10. The **interior** of biangle $\angle A'ABB'$ is

$$\mathring{\angle} A'ABB' := \mathring{\angle} A'AB \cap \mathring{\angle} ABB'.$$

If $P \in \mathring{\angle} A'ABB'$, either of rays $r(A, P), r(B, P)$ is called an **interior ray** of $\angle DABC$. If

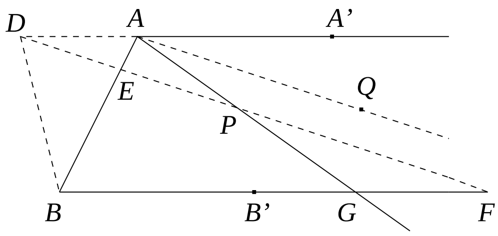

Figure 10: Biangle and limiting parallel

each interior ray $r(A, P)$ intersects $r(B, B')$, we say that $r(A, A')$ is **limiting parallel** to $r(B, B')$ and that biangle $\angle A'ABB'$ is **closed** at A, written $r(A, A') \upuparrows r(B, B')$.

Lemma 3. *Let $\angle A'ABB'$ be a biangle. See Figure 10.*

(a) If $D * A * A'$, then $r(D, A') \upuparrows r(B, B')$ if and only if $r(A, A') \upuparrows r(B, B')$.

(b) If $r(A, A') \upuparrows r(B, B')$, so is $r(B, B') \upuparrows r(A, A')$.

Proof. (a) Assume $r(D, A') \upuparrows r(B, B')$. Take a point P in the interior of $\angle A'ABB'$. It is clear that P is an interior point of biangle $\angle A'DBB'$. Then $r(D, P)$ meets $r(B, B')$ at F since $\angle A'DBB'$ is closed at D. Note that P is an interior point of $\angle BAF$. Then $r(A, P)$ is between $r(A, B)$ and $r(A, F)$. Thus $r(A, P)$ meets BF at G with $B * G * C$. By definition $r(A, A') \upuparrows r(B, B')$.

Conversely, assume $r(A, A') \upuparrows r(B, B')$. For each ray r between $r(D, B)$ and $r(D, A')$, we have r meeting AB at E between A and B. Pick a point P on r such that $D * E * P$. Note that $\angle A'AB > \angle AED \cong \angle BEP$. There is a ray $r(A, Q)$ such that $\angle BAQ \cong \angle BEP$. Then $r(A, Q) \| r(E, P)$. Since $r(A, Q)$ meets $r(B, B')$, we see that $r(E, P)$ must meet $r(B, B')$, i.e., $r(D, P)$ meet $r(B, B')$. Hence $\angle A'DBB'$ is closed at D.

(b) Given an interior point $P \in \mathring{\angle} ABB'$ and consider the ray $r(B, P)$. Suppose $r(B, P)$ does not meet $r(A, A')$. By the corollary of Aristotle's axiom there exists a point Q on $r(B, P)$ such that $\angle AQB < \angle PBB'$. See Figure 11. Note that $r(A, Q)$ meets $r(B, B')$ at C. Then we have triangle ΔBCQ. Thus $\angle AQB > \angle QBC = \angle PBB'$, which contradicts

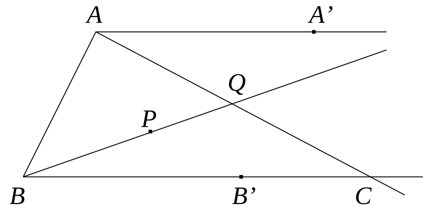

Figure 11: Biangle and limiting parallel

$\angle AQB < \angle PBB'$. So $r(B,P)$ must meet $r(A,A')$. Hence $\angle A'ABB'$ is closed at B, i.e., $r(B,B') \upuparrows r(A,A')$. $\square$

Proposition 4.2 (Transitivity of limiting parallelism)**.** *If both rays $r(A,A')$ and $r(B,B')$ are limiting parallel to ray $r(C,C')$, then $r(A,A')$ and $r(B,B')$ are limiting parallel to each other.*

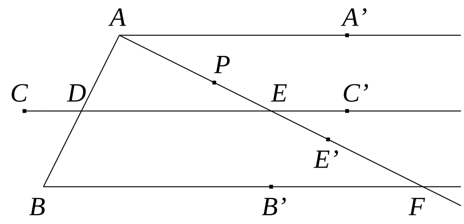

Figure 12: Limiting parallels

Proof. Case 1. Lines $\overline{AA'}$ and $\overline{BB'}$ are on opposite sides of line $\overline{CC'}$. See Figure 12.

It is clear that $r(A,A')$ and $r(B,B')$ have no point in common. Let AB meet $\overline{CC'}$ at D. We may assume $C * D * C'$. Now for each point P interior to $\angle A'AB$, the ray $r(A,P)$ meets $r(C,C')$ at E since $r(A,A') \upuparrows r(C,C')$. We may assume $C * E * C'$. Then $r(E,E')$ meets $r(B,B')$ at F since $r(C,C') \upuparrows r(B,B')$, where $E * E' * F$. Hence $r(A,A')$ meets $r(B,B')$ at F. Therefore by definition $r(A,A')$ and $r(B,B')$ are limiting parallel to each other.

Case 2. Lines $\overline{AA'}$ and $\overline{BB'}$ are on the same side of line $\overline{CC'}$.

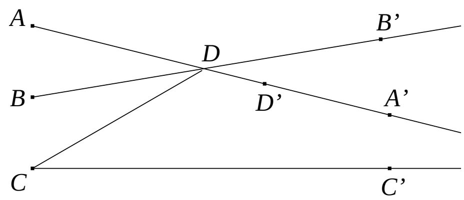

Figure 13: Limiting parallels

We first claim that $\overline{AA'}$ and $\overline{BB'}$ do not meet. Suppose $\overline{AA'}$ and $\overline{BB'}$ meet at point D. We may assume that D belongs to both rays $r(A,A'), r(B,B')$, and assume $A * D * A'$, $B * D * B'$. Take a point D' such that $A * D * D'$. Then $r(D,D')$ meets $r(C,C')$ since $r(D,B') \upuparrows r(C,C')$, i.e., $r(A,A')$ meets $r(C,C')$, which is a contradiction. See Figure 13.

Let AC meet $\overline{BB'}$ at point D. We may assume $B * D * B'$. For each point P interior to $\angle A'AC$, the ray $r(A,P)$ meets $r(C,C')$ at point E. Since $r(B,D)$ $(= r(B,B'))$ meets the triangle ΔDCE, the ray $r(B,B')$ meets either AE or CE. Since $r(B,B')$ does not meet $r(C,C')$, so $r(B,B')$ meet AE at F such that $A * F * E$. For point P interior to $\angle BAD$, the ray $r(A,P)$ meets BD between B and D, of course $r(A,P)$ meets $r(B,B')$. Hence $r(A,A')$ and $r(B,B')$ are limiting parallel to each other. $\square$

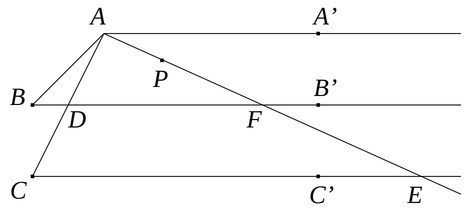

Figure 14: Limiting parallels

Two rays r, s are said to be **limiting parallel**, denoted $r \shortparallel s$, if $r \subset s$ or $s \subset r$ or $r \upuparrows s$. Then $\shortparallel$ is an equivalence relation on rays in hyperbolic geometry. An equivalence class of rays is called an **ideal point** or **end**, viewing it lying on each ray contained in the equivalence class. Since a point on a line separates the line into two opposite rays, and opposite rays are not equivalent, we see that every line has two ends on it.

If A, B are vertices of two rays r, s with $r \upuparrows s$. Let $\mathscr{R}$ denote the ideal point determined by these rays, i.e., $\mathscr{R} = [r] = [s]$. We write $r = A\mathscr{R}$ and $s = B\mathscr{R}$ and refer to the closed biangle with side r, s as a **singly asymptotic triangle** $\Delta AB\mathscr{R}$. We shall see that these triangles have some properties in common with ordinary triangles.

Lemma 4. *In hyperbolic geometry if two lines l, m are cut by a line t such that the alternate interior angles are congruent, then l, m are divergent parallels.*

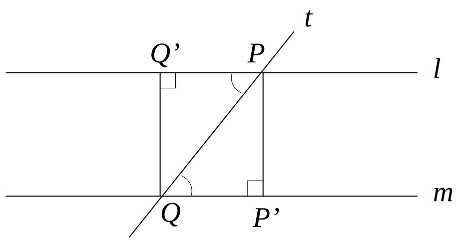

Figure 15: Asymptotic triangle

Proof. Let t meet l at P and meet m at Q such that $\angle Q'PQ \cong \angle P'QP$, where PP' is perpendicular to m with foot P' on m and QQ' is perpendicular to l with foot Q' on l. Then $\Delta PQQ' \cong \Delta QPP'$. So $PP' \cong QQ'$. Hence l, m are divergent parallel lines. $\square$

Proposition 4.3. *Let $\Delta AB\mathscr{R}$ be a singly asymptotic triangle with a single ideal point $\mathscr{R}$. Then the exterior angles at A, B are greater than their respective opposite interior angles, i.e., $\angle A < \operatorname{ext} \angle B$.*

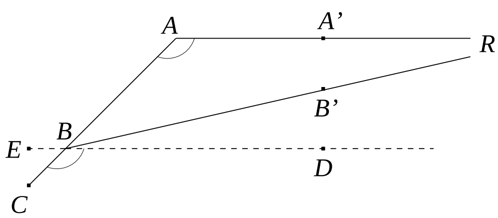

Figure 16: Asymptotic triangle

Proof. Extend AB to C such that $A * B * C$. Draw ray $r(B, D)$ such that $\angle CBD \cong \angle BAA'$ and extend DB to E such that $E * B * D$. Then $\angle ABE \cong \angle CBD \cong \angle BAA'$. Thus lines $\overline{BD}, \overline{AA'}$ are divergent parallels. Since $r(B, B') \upuparrows r(A, A')$, we see that $r(B, D) \neq r(B, B')$. If $r(B, D)$ is between $r(B, B')$ and $r(B, A)$, then $r(B, D)$ meets $r(A, A')$, which is a contradiction. So we must have $r(B, D)$ between $r(B, C)$ and $r(B, B')$. This means that $\angle CBD < \angle CBB'$, i.e., $\angle CBB' > \angle BAA'$. $\square$